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LONGITUDINAL–TRANSVERSE BENDING OF LAYERED BEAMS IN A THREE-DIMENSIONAL FORMULATION

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Three-dimensional equations of the elasticity theory for layered beams are solved by the method of asymptotic splitting without additional hypotheses or restrictions.

Key words: *multilayer structures, longitudinal–transverse bending.*

Introduction. The problem of calculation of multilayer structures is important because advanced technologies allow fabrication of structural elements made of dissimilar materials. Methods for calculating multilayer beams and plates have been rather adequately developed; the most significant works have been reviewed and analyzed, e.g., in [1, 2]. In most papers, special attention was paid to approximate allowance for cross-sectional shear in the plane of the load applied, transverse reduction of the layers and other types of shear being neglected. Appropriate three-dimensional equations were reduced to one- or two-dimensional equations by using piecewise-linear or power expansions of transverse shear stresses and displacements over the transverse coordinate. In many cases, however, the properties of materials can be so disparate that the remaining components of stress and strain tensors cannot be neglected. Materials differ from each other not only in rigidity but also in strength. For layered beams, therefore, it is necessary to have reliable information on stress fields in each layer. A method of asymptotic splitting, which does not require any significant assumptions or restrictions, is proposed in the present paper to solve three-dimensional equations of the elasticity theory for the case of bending of layered beams. This method was used previously to solve particular problems in [3–5].

1. Transverse Bending. Let us consider a beam with an arbitrary cross section constant over the length and symmetric with respect to the x axis; the beam consists of an arbitrary number of layers made of different materials (Fig. 1). The origin is on the upper surface of the beam; the layers are enumerated from top to bottom (i is the number of the current layer and s is the number of layers).

The quantities u, v, and w are the displacements of points in the x, y, and z directions, respectively, b_- and b_+ are the widths of the upper and lower surfaces of the beam, u^* is the characteristic value for the displacement u, l and h are the length and height of the beam, respectively, λ_i and ν_i are elastic constants, $\varepsilon_{\alpha\beta}$ are the components of the linear strain tensor, λ_0 is the characteristic value of the elastic constant, and q_- and q_+ are the transverse loads applied to the upper and lower surfaces of the beam, respectively. We consider only beams for which $\varepsilon = h/l$ is a small parameter. We use the following dimensionless variables and functions:

$$x' = \frac{x}{h}, \qquad y' = \frac{y}{h}, \qquad z' = \frac{z}{l}, \qquad u' = \frac{u}{u^*}, \qquad w' = \frac{w}{u^*}, \qquad v' = \frac{v}{u^*}, \qquad \lambda_i' = \frac{\lambda_i}{\lambda_0},$$

$$\mu_i' = \frac{\mu_i}{\lambda_0}, \qquad \sigma_{\alpha\beta}' = \frac{\sigma_{\alpha\beta}}{\sigma_0}, \qquad q_+' = \frac{q_+}{q_0}, \qquad q_-' = \frac{q_-}{q_0}, \qquad \sigma_0 = \frac{\lambda_0 u^*}{h}.$$

In what follows, the prime marking dimensionless quantities is omitted. We do not use the hypothesis of flat sections and use the following approximations for displacements in each layer:

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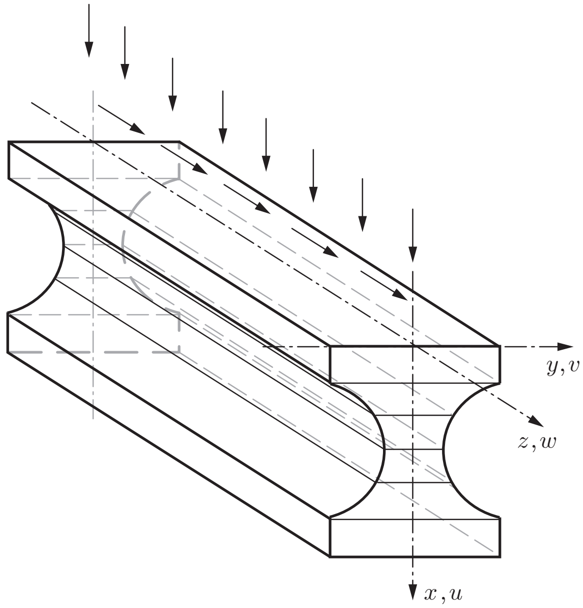

Fig. 1. Longitudinal–transverse bending of a layered beam.

$$w_i^{(n)}(\boldsymbol{r},\varepsilon)=\sum_{k=0}^{n}U_{i,k}^{z}\,\frac{d^{(2k+1)}u_0^{(n)}}{dz^{(2k+1)}}\,\varepsilon^{2k+1},\qquad U_{i,0}^{z}=-(x-c_0),\qquad U_{i,0}^{x}=1,$$

$$u_i^{(n)}(\boldsymbol{r},\varepsilon)=\sum_{k=0}^{n}U_{i,k}^{x}\,\frac{d^{2k}u_0^{(n)}}{dz^{2k}}\,\varepsilon^{2k},\qquad v_i^{(n)}(\boldsymbol{r},\varepsilon)=\sum_{k=1}^{n}U_{i,k}^{y}\,\frac{d^{2k}u_0^{(n)}}{dz^{2k}}\,\varepsilon^{2k}. \tag{1}$$

Here $\boldsymbol{r}$ is the radius vector of a point, $u_i^{(n)}(\boldsymbol{r},\varepsilon)$ is the approximation of the displacement vector, c_0 is a constant, n is the ordinal number of the approximation, and $U_k^z(x,y)$, $U_k^x(x,y)$, and $U_k^y(x,y)$ are the characteristic functions of the vector field of displacements in the beam cross section. We define the bending function $u_0^{(n)}(z)$ as

$$u_0^{(n)}(z)=\frac{1}{F}\sum_{i=1}^{s}\int\limits_{F_i}u_i^{(n)}(\boldsymbol{r})\,dF,\qquad \sum_{i=1}^{s}\int\limits_{F_i}U_{i,k}^{x}(x,y)\,dF=0\qquad (k=1,\ldots,n),$$

where F is the cross-sectional area of the beam and F_i is the area of the ith layer of the beam cross section.

The beam material obeys Hooke's law:

$$(\sigma_{\alpha\beta})_i=\lambda_i\theta\delta_{\alpha\beta}+2\mu_i\varepsilon_{\alpha\beta},\quad \theta=\sum_{\gamma=1}^{3}\varepsilon_{\gamma\gamma},\quad \lambda_i=\frac{\nu_iE_i}{(1-2\nu_i)(1+\nu_i)},\quad \mu_i=\frac{E_i}{2(1+\nu_i)}. \tag{2}$$

We substitute equalities (1) into Hooke's law (2):

$$(\sigma_{\alpha\alpha})_i^{(n)}=\sum_{k=1}^{n}(\tau_{\alpha\alpha})_i^{(2k)}\,\frac{d^{2k}u_0^{(n)}}{dz^{2k}}\,\varepsilon^{2k}+(\lambda_i+2\mu_i\delta_{\alpha z})U_{i,n}^{z}\,\frac{d^{2n+2}u_0^{(n)}}{dz^{2n+2}}\,\varepsilon^{2n+2},\qquad \alpha\in[x,y,z],$$

$$(\sigma_{xy})_i^{(n)}=\sum_{k=1}^{n}(\tau_{xy})_i^{(2k)}\,\frac{d^{2k}u_0^{(n)}}{dz^{2k}}\,\varepsilon^{2k},\quad (\sigma_{\beta z})_i^{(n)}=\sum_{k=1}^{n}(\tau_{\beta z})_i^{(2k+1)}\,\frac{d^{2k+1}u_0^{(n)}}{dz^{2k}}\,\varepsilon^{2k+1},\quad \beta\in[x,y] \tag{3}$$

($\delta_{\alpha z}$ is the Kronecker delta). Formulas (3) contain characteristic functions of the tensor fields of stresses in the beam cross section $(\tau_{\alpha\beta})_i^{(j)}$, which are related to the characteristic functions of the vector field of displacements as follows:

$$(\tau_{zz})_i^{(2k+2)}=(\lambda_i+2\mu_i)U_{i,k}^{z}+\lambda_i\Big(\frac{\partial U_{i,k+1}^{x}}{\partial x}+\frac{\partial U_{i,k+1}^{y}}{\partial y}\Big),\qquad (\tau_{xz})_i^{(1)}=0,\qquad (\tau_{yz})_i^{(1)}=0,$$

$$(\tau_{xy})_i^{(2k)} = \mu_i\Big(\frac{\partial U_{i,k}^y}{\partial x} + \frac{\partial U_{i,k}^x}{\partial y}\Big), \qquad (\tau_{\beta z})_i^{(2k+1)} = \mu_i\Big(U_{i,k}^\beta + \frac{\partial U_{i,k}^z}{\partial \beta}\Big), \qquad \beta,\gamma \in [x,y], \quad \gamma \ne \beta,$$

$$(\tau_{\beta\beta})_i^{(2k+2)} = \Big(\lambda_i\Big(U_{i,k}^z + \frac{\partial U_{i,k+1}^\gamma}{\partial \gamma}\Big) + (\lambda_i + 2\mu_i)\,\frac{\partial U_{i,k+1}^\beta}{\partial \beta}\Big), \qquad i = 1,\ldots,s, \qquad k = 1,\ldots,n.$$

We assume that the transverse loads on the upper and lower surfaces have the form

$$q_- = \sum_{k=1}^n q_-^{(2k)}\,\frac{d^{2k}u_0^{(n)}}{dz^{2k}}\,\varepsilon^{2k}, \qquad q_+ = \sum_{k=1}^n q_+^{(2k)}\,\frac{d^{2k}u_0^{(n)}}{dz^{2k}}\,\varepsilon^{2k}, \qquad q_-^{(2)} = q_+^{(2)} = 0,$$
$$q = b_-q_- + b_+q_+, \qquad q = \sum_{k=1}^n q^{(2k)}\,\frac{d^{2k}u_0^{(n)}}{dz^{2k}}\,\varepsilon^{2k}, \tag{4}$$

where $q_-^{(2k)}$, $q_+^{(2k)}$, and $q^{(2k)}$ $(k = 2,\ldots,n)$ are constants and q is the total transverse load.

We write the equilibrium equations in the form

$$L_{x,i}(\boldsymbol{u},\varepsilon) = 0, \qquad L_{y,i}(\boldsymbol{u},\varepsilon) = 0, \qquad L_{z,i}(\boldsymbol{u},\varepsilon) = 0. \tag{5}$$

The following boundary conditions are used:

— on the upper surface for $x = 0$,

$$J_{x,1}(\boldsymbol{u},\varepsilon) = 0, \qquad J_{y,1}(\boldsymbol{u},\varepsilon) = 0, \qquad J_{z,1}(\boldsymbol{u},\varepsilon) = 0;$$

— on the lower surface for $x = 1$,

$$J_{x,s}(\boldsymbol{u},\varepsilon) = 0, \qquad J_{y,s}(\boldsymbol{u},\varepsilon) = 0, \qquad J_{z,s}(\boldsymbol{u},\varepsilon) = 0; \tag{6}$$

— on the side surface,

$$B_{x,i}(\boldsymbol{u},\varepsilon) = 0, \qquad B_{y,i}(\boldsymbol{u},\varepsilon) = 0, \qquad B_{z,i}(\boldsymbol{u},\varepsilon) = 0.$$

We use the following conjugation conditions at the interface between the beam layers:

$$\Phi_{x,i}(\boldsymbol{u},\varepsilon) = 0, \qquad \Phi_{y,i}(\boldsymbol{u},\varepsilon) = 0, \qquad \Phi_{z,i}(\boldsymbol{u},\varepsilon) = 0,$$
$$S_{x,i}(\boldsymbol{u},\varepsilon) = 0, \qquad S_{y,i}(\boldsymbol{u},\varepsilon) = 0, \qquad S_{z,i}(\boldsymbol{u},\varepsilon) = 0, \qquad x = h_i, \qquad i = 2,\ldots,s. \tag{7}$$

We use the following differential operators acting on the displacement vector $\boldsymbol{u}$ in Eqs. (5)–(7):

$$L_{\beta,i}(\boldsymbol{u},\varepsilon) = \frac{\partial\,(\sigma_{\beta x})_i}{\partial x} + \frac{\partial\,(\sigma_{\beta y})_i}{\partial y} + \varepsilon\,\frac{\partial\,(\sigma_{\beta z})_i}{\partial z}, \qquad J_{x,1}(\boldsymbol{u},\varepsilon) = (\sigma_{xx})_1 + q_-,$$

$$J_{y,1}(\boldsymbol{u},\varepsilon) = (\sigma_{xy})_1, \quad J_{z,1}(\boldsymbol{u},\varepsilon) = (\sigma_{xz})_1, \quad J_{x,s}(\boldsymbol{u},\varepsilon) = (\sigma_{xx}) - q_+, \quad J_{y,s}(\boldsymbol{u},\varepsilon) = (\sigma_{xy})_s,$$

$$J_{z,s}(\boldsymbol{u},\varepsilon) = (\sigma_{xz})_s, \quad B_{\beta,i}(\boldsymbol{u},\varepsilon) = (\sigma_{\beta x})_i n_x + (\sigma_{\beta y})_i n_y, \quad \Phi_{x,i}(\boldsymbol{u},\varepsilon) = (u)_{i-1} - (u)_i,$$

$$\Phi_{y,i}(\boldsymbol{u},\varepsilon) = (v)_{i-1} - (v)_i, \quad \Phi_{z,i}(\boldsymbol{u},\varepsilon) = (w)_{i-1} - (w)_i, \quad S_{x,i}(\boldsymbol{u},\varepsilon) = (\sigma_{xx})_{i-1} - (\sigma_{xx})_i,$$

$$S_{y,i}(\boldsymbol{u},\varepsilon) = (\sigma_{xy})_{i-1} - (\sigma_{xy})_i, \quad S_{z,i}(\boldsymbol{u},\varepsilon) = (\sigma_{zx})_{i-1} - (\sigma_{zx})_i, \quad \beta \in [x,y,z].$$

Definition 1. The problem of finding the displacement field $\boldsymbol{u}$ satisfying Eqs. (2) and (5)–(7) inside the beam and at its boundary, will be called the semi-boundary problem because there are segments of the beam boundary (its end faces) where no boundary conditions are temporarily imposed.

Definition 2. Let the differential equation $L(\boldsymbol{u}(\boldsymbol{r}),\varepsilon) = 0$ be given. The functional sequence $\{\boldsymbol{u}^{(n)}(\boldsymbol{r})\}_{n=1}^\infty$ will be called the formal asymptotic solution of this equation if there exists a monotonically increasing function $m(n)$ such that the equality $L(\boldsymbol{u}^{(n)}(\boldsymbol{r}),\varepsilon) = O(\varepsilon^{m(n)})$ is satisfied for all n as $\varepsilon \to 0$. If a similar equality is satisfied for some boundary conditions or all of them, we will speak about the formal asymptotic solution of the semi-boundary or boundary problem, respectively.

We require satisfaction of equalities for the characteristic functions of the tensor field of stresses and the associated characteristic functions of the vector field of displacements:

— inside the cross section,

$$\frac{\partial\,(\tau_{\beta x})_i^{(2k)}}{\partial x} + \frac{\partial\,(\tau_{\beta y})_i^{(2k)}}{\partial y} + (\tau_{\beta z})_i^{(2k-1)} = 0, \quad \frac{\partial\,(\tau_{zx})_i^{(2k+1)}}{\partial x} + \frac{\partial(\tau_{zy})_i^{(2k+1)}}{\partial y} + (\tau_{zz})_i^{(2k)} = 0; \tag{8}$$

— on the upper and lower surfaces of the beam,

$$\begin{gathered}(\tau_{zx})_1^{(2k+1)} = 0, \quad (\tau_{xy})_1^{(2k)} = 0, \quad (\tau_{xx})_1^{(2k)} = -q_-^{(2k)}, \quad k = 1, \ldots, n, \qquad x = 0, \\ (\tau_{xx})_s^{(2k)} = q_+^{(2k)}, \quad (\tau_{zx})_s^{(2k+1)} = 0, \quad (\tau_{xy})_s^{(2k)} = 0, \quad k = 1, \ldots, n, \quad x = 1;\end{gathered} \tag{9}$$

— on the side surface of the beam,

$$(\tau_{\beta x})_i^{(2k)} n_x + (\tau_{\beta y})_i^{(2k)} n_y = 0, \quad \beta \in [x, y], \qquad (\tau_{zy})_i^{(2k+1)} n_y + (\tau_{zx})_i^{(2k+1)} n_x = 0; \tag{10}$$

— at the interface between the beam layers,

$$\begin{gathered}(\tau_{zx})_{i-1}^{(2k+1)} = (\tau_{zx})_i^{(2k+1)}, \qquad (\tau_{x\beta})_{i-1}^{(2k)} = (\tau_{x\beta})_i^{(2k)}, \qquad U_{i-1,k}^{\alpha} = U_{i,k}^{\alpha}, \\ \alpha \in [x, y, z], \qquad \beta \in [x, y], \qquad x = h_i, \qquad i = 2, \ldots, s, \qquad k = 1, \ldots, n.\end{gathered} \tag{11}$$

For the first two differential equalities in (4) to be simultaneously satisfied, it suffices to require proportionality of the upper and lower loads:

$$q_+^{(2k)} = k_q q_-^{(2k)}, \qquad q_+ = k_q q_-. \tag{12}$$

Integrating Eqs. (8) over the cross section with allowance for Eqs. (9)–(11), we obtain the necessary condition for solvability of the boundary problem (8)–(11):

$$q^{(2k)} = -I_{(2k-2)}, \qquad I_{(2k)} = \int\limits_F (x - c_0)(\tau_{zz})^{(2k)}\, dF. \tag{13}$$

Summing up the first two equalities in (4) multiplied by b_- and b_+, respectively, we obtain a differential equation equivalent to the initial equalities (4) because of load proportionality:

$$\sum_{k=2}^{n} I_{(2k-2)} \frac{d^{2k} u_0^{(n)}}{dz^{2k}} \varepsilon^{2k} + q = 0. \tag{14}$$

By forward substitution, we can readily verify that formulas (1) yield a formal asymptotic solution of the semi-boundary problem (5)–(7) if the condition $\varepsilon^4 \, d^{2n+2} u_0^{(n)}/dz^{2n+2} = O(1)$ is satisfied as $\varepsilon \to 0$. The latter equality is reached if we eliminate rapidly oscillating solutions as $\varepsilon \to 0$ in solving the differential equation (14). Actually, this condition means identification of a four-parameter family of solutions of Eq. (14):

$$u_0^{(n)}(z, \varepsilon) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + F_{(4)}(z, \varepsilon). \tag{15}$$

Here a_j are parameters and $F_{(4)}(z, \varepsilon)$ is a particular "nonoscillating" solution of the equation.

The total number of boundary conditions for the family of solutions (15) at the end faces of the beam is four. Hence, traditional boundary conditions based on the Saint Venant principle can be used at the end faces: zero mean displacements $u_0^{(n)}$ and bending moment (simply supported beam); zero bending moment and shear force (free end face); zero longitudinal and transverse displacements on the average (clamped beam).

If the transverse load is a polynomial, we can use forward substitution to prove the following statement.

Statement 1 (about the exact solution of the semi-boundary problem)**.** Let conditions (12) and (13) be satisfied, formula (15) be valid, and the transverse load $q(z)$ be a polynomial of power m_0. Then, functions (1) yield a four-parameter family of exact solutions of the semi-boundary problem (2), (5)–(7). The number of approximation n is calculated by the formula $n = [0.5(m_0 + 4)]$, where $[a]$ is the integer part of the number a.

Example 1 (one-layer beam with a rectangular cross section under the action of point forces)**.** If the beam experiences the action of point forces only, in accordance with the statement given above, the second approximation ($n = 2$) yields the exact solution, and Eq. (14) yields the equalities

$$\frac{d^j u_0(z)}{dz^j} = 0, \qquad j \geqslant 4.$$

TABLE 1

b	y	$\Delta\sigma_{zx}$	b	y	$\Delta\sigma_{zx}$
0.5	0 $0.5b$	0.983 1.033	2	0 $0.5b$	0.856 1.396
1	0 $0.5b$	0.940 1.126	4	0 $0.5b$	0.805 1.987

TABLE 2

Type of beam support	Δu						
	$\varepsilon = 0.1$		$\varepsilon = 0.125$		$\varepsilon = 0.167$		$\varepsilon = 0.25$
	$b = 1$	$b = 4$	$b = 1$	$b = 4$	$b = 1$	$b = 4$	$b = 1$
Cantilever	1.007	1.021	1.012	1.033	1.021	1.060	1.046
Simply supported	1.030	1.086	1.046	1.134	1.082	1.236	1.184
Clamped at the ends	1.118	1.343	1.184	1.536	1.329	1.957	1.738

If we substitute these equalities into formulas (1) and (3), already the first approximation yields the exact solution of the semi-boundary problem (2), (5)–(7). By solving the boundary problem (8)–(11) with $k = 1$, we obtain the following characteristic functions of the stress tensor for a one-layer rectangular-section beam (b is the beam width):

$$(\tau_{xx})^{(2)} = (\tau_{xy})^{(2)} = 0, \qquad (\tau_{zz})^{(2)} = -E(x - c_0),$$

$$(\tau_{zy})^{(3)} = 4\mu\nu b \sum_{k=1}^{\infty} \frac{\sinh\,[(2k-1)\pi y]}{\sinh\,[(2k-1)\pi 0.5b]} \, \frac{\cos((2k-1)\pi x)}{\pi^2(2k-1)^2} + 2\nu\mu y(x - 0.5) + E(0.5(x - 0.5)^2 - 0.125). \tag{16}$$

Using (16), we calculate the displacements and stress-tensor components by formulas (1) and (3). If we average $(\sigma_{zx})^{(1)}$ over the cross-section width, formulas (3) and (16) yield the known Zhuravskii's formula for the cross-sectional distribution of shear stresses.

We introduce the ratio of shear stresses to their averaged values:

$$\Delta\sigma_{zx} = \frac{(\sigma_{zx})^{(1)}}{\langle(\sigma_{zx})^{(1)}\rangle} = \frac{(\tau_{zx})^{(3)}}{\langle(\tau_{zx})^{(3)}\rangle} \qquad \Bigl(\langle a\rangle = \frac{1}{b}\int_{-0.5b}^{0.5b} a(x, y)\, dy\Bigr).$$

The maximum values of $\Delta\sigma_{zx}$ in the beam cross section for $x = 0.5$ and $\nu = 0.25$ are listed in Table 1. It follows from the definition of this quantity that it is independent of the method of beam attachment and the number of point forces applied. The values in Table 1 coincide with the values obtained by Timoshenko for a rectangular-section cantilever beam [6] loaded by a point force at the end (Saint Venant problem). Hence, Timoshenko's result is extended to arbitrarily supported beams with an arbitrary number of point loads.

We introduce the ratio of the maximum values of bending obtained by the method proposed to the maximum bending obtained on the basis of Bernoulli's hypothesis of flat sections $u_{\rm B}$:

$$\Delta u = u_0^{(1)}/u_{\rm B}.$$

The values of Δu for beams under the action of a unit point load for $\nu = 0.25$ are listed in Table 2. In all examples considered, this ratio is greater than unity.

It should be noted that the action of point forces applied away from the end faces is manifested in cutting the beam over the section where the point load is acting. Then, these two parts are conjugated with the use of integral conditions at the end faces, which is possible owing to the Saint Venant principle. Therefore, there is always an error near the point of action of the point force, which is typical of the beam theory. This error can be taken into account and corrected only on the basis of constructing boundary layers [7].

Example 2 (plane deformation of a three-layer beam)**.** Let us consider a three-layer beam of unit width ($b = 1$) loaded at the upper surface by a transverse distributed load q_- ($k_q = 0$). We assume that the layers have the following parameters: $E_2 = 1$, $E_1 = E_3 = 4E_2$, $\nu_1 = \nu_2 = \nu_3 = 0.3$, $h_2 = 0.33$, and $h_3 = 0.67$. The characteristic functions of the displacement vector and stress tensor are shown in Figs. 2 and 3. The stresses σ_{zz} are an order of magnitude greater than the stresses σ_{xx}, as it follows from the diagrams of the characteristic functions τ_{zz} and τ_{xx}.

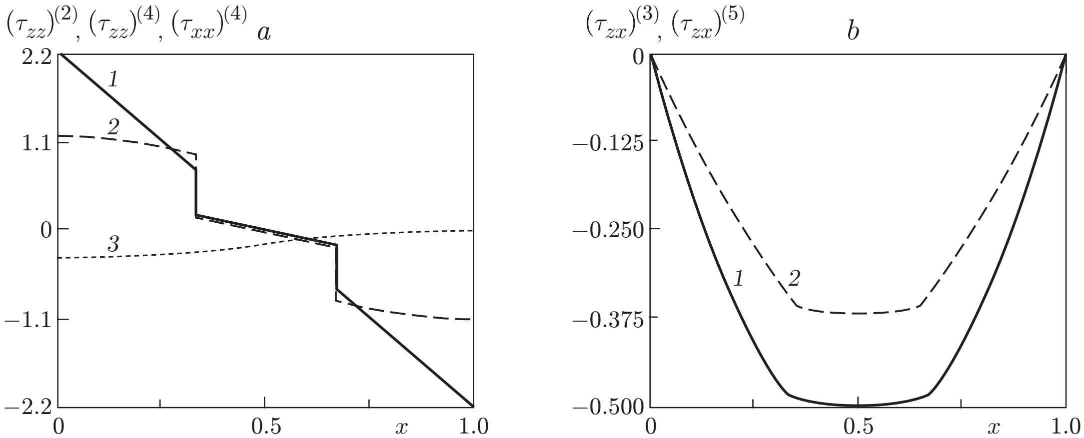

Fig. 2. Characteristic functions of the stress tensor for the three-layer beam ($E_1 = E_3 = 4E_2$ and $\nu_1 = \nu_2 = \nu_3 = 0.3$): (a) normal functions $(\tau_{zz})^{(2)}$ (1), $(\tau_{zz})^{(4)}$ (2), and $(\tau_{xx})^{(4)}$) (3); (b) tangential functions $(\tau_{zx})^{(3)}$ (1) and $(\tau_{zx})^{(5)}$ (2).

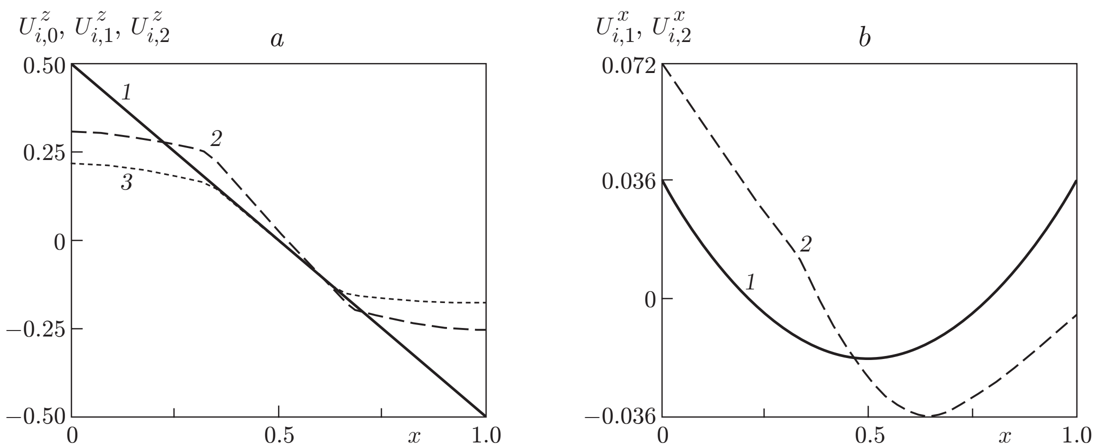

Fig. 3. Characteristic functions of the displacement vector for the three-layer beam ($E_1 = E_3 = 4E_2$ and $\nu_1 = \nu_2 = \nu_3 = 0.3$): (a) longitudinal functions $U^z_{i,0}$ (1), $U^z_{i,1}$ (2), and $U^z_{i,2}$ (3); (b) transverse functions $U^x_{i,1}$ (1) and $U^x_{i,2}$ (2).

The solutions obtained on the basis of the above-formulated statement for one- and three-layer cantilever beams coincide with the solutions obtained with the use of the Airy functions (see [6, 8]).

Thus, reduction can be neglected in the present case. In the general case of calculating layered beams, however, it is impossible to *a priori* neglect these or those components of the stress tensor because their values can depend on the geometric size of the layers, mechanical characteristics of structural materials, and positioning of materials in the structure.

Example 3 (plane deformation of a two-layer cantilever beam)**.** Let us consider a two-layer cantilever beam of unit width ($b = 1$) loaded at the upper surface by a constant transverse distributed load q_- ($k_q = 0$). We assume that the layers have the following characteristics: the lower layer is steel ($E_2 = 200$ GPa and $\nu_1 = 0.33$); the upper layer is graphite ($E_1 = 5.9$ GPa and $\nu_1 = 0.3$) or concrete ($E_1 = 20$ GPa and $\nu_1 = 0.2$). The upper layer thickness takes the values $\Delta h_1 = 0.05$, 0.1, and 0.2. Table 3 contains the values of $|\sigma_{zz}/\sigma_{xx}|$ calculated in the origin (upper point of the built-in section) where the axial tensile stresses σ_{zz} in the upper layer have the maximum values. Obviously, if this ratio is close to unity or smaller than unity, reduction cannot be neglected. It follows from the analysis of Table 3 that the necessity of taking reduction into account depends on the properties of materials, relative thicknesses of the layers Δh_1, and relative longitudinal size of the beam ε. The axial stresses decrease with

TABLE 3

ε	$\lvert\sigma_{zz}/\sigma_{xx}\rvert$					
	$\Delta h_1 = 0.05$		$\Delta h_1 = 0.1$		$\Delta h_1 = 0.2$	
	Graphite	Concrete	Graphite	Concrete	Graphite	Concrete
0.100	0.5	2.6	0.7	3.2	1.3	4.8
0.125	0.2	1.5	0.3	1.9	0.7	2.9
0.167	0.1	0.7	0.03	0.9	0.2	1.5

distance from the built-in face, whereas the value of σ_{xx} remains unchanged; hence, the relative contribution of reduction increases.

2. Longitudinal–Transverse Bending. Let the upper and lower surfaces of the beam be affected, in addition to the transverse load, by a longitudinal distributed load p_-, p_+. We use the following approximations for the displacements in each layer:

$$w_i^{(n)} = \sum_{k=0}^{n} W_{i,k}^z \frac{d^{2k} w_0^{(n)}}{dz^{2k}} \varepsilon^{2k}, \qquad u_i^{(n)} = \sum_{k=1}^{n} W_{i,k}^x \frac{d^{(2k-1)} w_0^{(n)}}{dz^{(2k-1)}} \varepsilon^{2k-1}, \qquad W_{i,0}^z = 1,$$
$$v_i^{(n)} = \sum_{k=1}^{n} W_{i,k}^y \frac{d^{(2k-1)} w_0^{(n)}}{dz^{(2k-1)}} \varepsilon^{2k-1}, \qquad \boldsymbol{u}_i^{(n)}(\boldsymbol{r}) = (u_i^{(n)}, v_i^{(n)}, w_i^{(n)}). \tag{17}$$

Here $w_0^{(n)}(z)$ is the function of the longitudinal section of displacements and $W_{i,k}^z(x,y)$, $W_{i,k}^x(x,y)$, and $W_{i,k}^y(x,y)$ are the characteristic functions of the displacement vector in the beam cross section. We assume that the value of the longitudinal displacement function $w_0^{(n)}(z)$ equals the mean displacement of cross-sectional points in the longitudinal direction, which corresponds to the equalities

$$\sum_{i=1}^{s} \int_{F_i} W_{i,k}^z(x,y)\, dF = 0, \qquad k = 1,\ldots,n.$$

Using the expressions for displacements (17) in Hooke's law, we obtain

$$(\sigma_{\alpha\alpha})_i^{(n)} = \sum_{k=1}^{n} (\tau_{\alpha\alpha})_i^{(2k-1)} \frac{d^{2k-1} w_0^{(n)}}{dz^{2k-1}} \varepsilon^{2k-1} + (\lambda_i + 2\mu_i \delta_{\alpha z}) W_{i,n}^z \frac{d^{2n+1} w_0^{(n)}}{dz^{2n+1}} \varepsilon^{2n+1},$$
$$(\sigma_{xy})_i^{(n)} = \sum_{k=1}^{n} (\tau_{xy})_i^{(2k-1)} \frac{d^{2k-1} w_0^{(n)}}{dz^{2k-1}} \varepsilon^{2k-1}, \qquad (\sigma_{\beta z})_i^{(n)} = \sum_{k=1}^{n} (\tau_{\beta z})_i^{(2k)} \frac{d^{2k} w_0^{(n)}}{dz^{2k}} \varepsilon^{2k}, \tag{18}$$
$$\alpha \in [x,y,z], \qquad \beta \in [x,y].$$

In formulas (18), we used the characteristic functions of the stress tensor $(\tau_{\alpha\beta})_i^{(j)}$, which are related to the characteristic functions of the displacement vector:

$$(\tau_{zz})_i^{(2k-1)} = (\lambda_i + 2\mu_i) W_{i,k-1}^z + \lambda_i \Big(\frac{\partial W_{i,k}^x}{\partial x} + \frac{\partial W_{i,k}^y}{\partial y} \Big), \qquad (\tau_{z\beta})_i^{(2k)} = \mu_i \Big(\frac{\partial W_{i,k}^z}{\partial \beta} + W_{i,k}^\beta \Big),$$
$$(\tau_{\beta\beta})_i^{(2k+1)} = \Big(\lambda_i \Big(W_{i,k}^z + \frac{\partial W_{i,k+1}^\gamma}{\partial \gamma} \Big) + (\lambda_i + 2\mu_i) \frac{\partial W_{i,k+1}^\beta}{\partial \beta} \Big), \qquad (\tau_{z\beta})_i^0 = 0,$$
$$(\tau_{xy})_i^{(2k-1)} = \mu_i \Big(\frac{\partial W_{i,k}^y}{\partial x} + \frac{\partial W_{i,k}^x}{\partial y} \Big), \quad \beta, \gamma \in [x,y], \quad \gamma \neq \beta, \quad i = 1,\ldots,s, \quad k = 1,\ldots,n.$$

We assume that the loads on the upper and lower surfaces have the form

$$p_- = \sum_{k=1}^{n} p_-^{(2k)} \frac{d^{2k} w_0^{(n)}}{dz^{2k}} \varepsilon^{2k}, \quad p_+ = \sum_{k=1}^{n} p_+^{(2k)} \frac{d^{2k} w_0^{(n)}}{dz^{2k}} \varepsilon^{2k}, \quad p = b_- p_- + b_+ p_+,$$

$$q_- = \sum_{k=1}^{n} q_-^{(2k-1)} \frac{d^{2k-1} w_0^{(n)}}{dz^{2k-1}} \varepsilon^{2k-1}, \quad q_+ = \sum_{k=1}^{n} q_+^{(2k-1)} \frac{d^{2k-1} w_0^{(n)}}{dz^{2k-1}} \varepsilon^{2k-1}, \quad q = b_- q_- + b_+ q_+. \tag{19}$$

In the differential operators (7), we take into account the influence of the longitudinal load:

$$J_{z,1}(\boldsymbol{u}, \varepsilon) = (\sigma_{xz})_1 + p_-, \qquad J_{z,s}(\boldsymbol{u}, \varepsilon) = (\sigma_{xz})_s - p_+. \tag{20}$$

Then, the semi-boundary problem (2), (5)–(7) is also valid for longitudinal–transverse bending. We require satisfaction of the following equalities for the characteristic functions of the stress tensor and associated characteristic functions of the displacement vector:

— on the upper and lower surfaces of the beam,

$$\begin{gathered} (\tau_{zx})_1^{(2k)} = -p_-^{(2k)}, \qquad (\tau_{xy})_1^{(2k-1)} = 0, \qquad (\tau_{xx})_1^{(2k-1)} = -q_-^{(2k-1)} \qquad \text{for} \quad x = 0, \\ (\tau_{zx})_s^{(2k)} = p_+^{(2k)}, \qquad (\tau_{xy})_s^{(2k-1)} = 0, \qquad (\tau_{xx})_s^{(2k-1)} = q_+^{(2k-1)} \qquad \text{for} \quad x = 1; \end{gathered} \tag{21}$$

— on the side surface of the beam,

$$(\tau_{\beta x})_i^{(2k-1)} n_x + (\tau_{\beta y})_i^{(2k-1)} n_y = 0, \qquad (\tau_{zy})_i^{(2k)} n_y + (\tau_{zx})_i^{(2k)} n_x = 0, \qquad k = 1, \ldots, n; \tag{22}$$

— on the interfaces between the layers,

$$(\tau_{zx})_{i-1}^{(2k)} = (\tau_{zx})_i^{(2k)}, \qquad (\tau_{x\beta})_{i-1}^{(2k-1)} = (\tau_{x\beta})_i^{(2k-1)}, \qquad W_{i-1,k}^{\alpha} = W_{i,k}^{\alpha}; \tag{23}$$

— at internal points of the beam cross section,

$$\begin{gathered} \frac{\partial\, (\tau_{\beta x})_i^{(2k-1)}}{\partial x} + \frac{\partial\, (\tau_{\beta y})_i^{(2k-1)}}{\partial y} + (\tau_{\beta z})_i^{(2k-2)} = 0, \quad \frac{\partial (\tau_{zx})_i^{(2k)}}{\partial x} + \frac{\partial (\tau_{zy})_i^{(2k)}}{\partial y} + (\tau_{zz})_i^{(2k-1)} = 0, \\ \alpha \in [x, y, z], \qquad \beta \in [x, y], \qquad k = 1, \ldots, n, \qquad i = 1, \ldots, s. \end{gathered} \tag{24}$$

For four differential equalities in (19) to be simultaneously satisfied, it suffices to require proportionality of the upper and lower loads:

$$q_+^{(2k-1)} = k_q q_-^{(2k-1)}, \qquad q_+ = k_q q_-, \qquad p_+^{(2k)} = k_p p_-^{(2k)}.$$

If we integrate Eqs. (24) over the cross section with allowance for equalities (21)–(23), we obtain the necessary condition of solvability of the boundary problem (21)–(24):

$$\begin{gathered} q_+^{(2k-1)} = -\frac{G_{(2k-2)}}{b_+ + k_q b_-}, \qquad p_+^{(2k)} = -\frac{A_{(2k-1)}}{b_+ + k_p b_-}, \\ G_{(2k)} = \sum_{i=1}^{s} \int_{F_i} (\tau_{xz})_i^{(2k)}\, dF, \qquad A_{(2k-1)} = \sum_{i=1}^{s} \int_{F_i} (\tau_{zz})_i^{(2k-1)}\, dF, \qquad k = 1, \ldots, n. \end{gathered} \tag{25}$$

A linear combination of the first two equalities in (19) yields the differential equation on the longitudinal displacement function

$$\sum_{k=1}^{n} A_{(2k-1)} \frac{d^{2k} w_0^{(n)}}{dz^{2k}} \varepsilon^{2k} + p = 0. \tag{26}$$

From Eqs. (19) and (25), there follows the equality to be satisfied by the transverse load:

$$q = -\sum_{k=2}^{n} G_{(2k-2)} \frac{d^{2k-1} w_0^{(n)}}{dz^{2k-1}} \varepsilon^{2k-1}. \tag{27}$$

Hence, the total cross-sectional load is not an arbitrarily prescribed function, in contrast to the total longitudinal load over the section. We denote this transverse load depending on the longitudinal load as q^p.

By forward substitution, we can easily verify that Eqs. (17) define a formal asymptotic solution of the semi-boundary problem (2), (5)–(7), (20) if the condition $\varepsilon^2\, d^{2n+1} w_0^{(n)} / dz^{2n+1} = O(1)$ is satisfied as $\varepsilon \to 0$. The latter equality is reached by eliminating rapidly oscillating solutions in solving the differential equation (26) as $\varepsilon \to 0$. Actually, this condition means identification of a two-parameter family of solutions of Eq. (26):

$$w_0^{(n)}(z, \varepsilon) = a_0 + a_1 z + F_{(2)}(z, \varepsilon). \tag{28}$$

Here a_j are parameters and $F_{(2)}(z, \varepsilon)$ is a particular "nonoscillating" solution of the equation. The total number of the boundary conditions for solutions (28) at the end faces of the beam is two. Hence, traditional boundary conditions based on the Saint Venant principle should be used at the beam ends: zero mean displacements $w_0^{(n)}$ (fixed end face) and zero longitudinal force (free end face).

3. General Case of the Longitudinal–Transverse Bending. Let the beam experience arbitrary longitudinal and transverse loads p and q simultaneously. The solution cannot be found in the form (17) because the loads are arbitrary. The load applied can be considered as a superposition of two loads. The first load is the longitudinal–transverse load of a special type: the longitudinal load p is arbitrary, and the transverse load q^p has the form (27). The second load is a reduced transverse load calculated by the formula $q^h = q - q^p$. This is the case of purely transverse bending, and it is resolved on the basis of representation (1). Because of the linearity of equations of the elasticity theory and their corollaries, the solutions obtained for the first and second loads are summed up.

REFERENCES

1. V. V. Bolotin, "Strength, stability, and oscillations of multilayer plates," in: *Strength Calculations* (collected scientific papers) [in Russian], No. 11, Mashinostroenie, Moscow (1965), pp. 31–63.
2. A. N. Andreev and Yu. V. Nemirovskii, *Multilayer Anisotropic Shells and Plates: Bending, Stability, and Oscillations* [in Russian], Nauka, Novosibirsk (2001).
3. G. L. Gorynin and Yu. V. Nemirovskii, "Calculations of the stress state of layered beams with an arbitrary cross-sectional shape," in: *Basic and Applied Problems of Modern Mechanics*, Proc. III All-Russia Conf. (Tomsk, October 2–4, 2002), Izd. Tomsk. Univ., Tomsk (2002), pp. 16–22.
4. Yu. V. Nemirovskii and G. L. Gorynin, "Asymptotic solution of the boundary problem of bending of extended elastic bodies," in: *Boundary Problems and Mathematical Simulation*, Proc. 5th All-Russia Conf. (Novokuznetsk, November 29–December 1, 2002), Izd. Novokuzn. Fil. Kemer. Univ., Novokuznetsk (2002), pp. 48–54.
5. U. V. Nemirovsky and G. L. Gorynin, "The theory of the layered beams under the action of cross loading," in: *Advanced Studies in Mechanical Engineering*, Yeungnam Univ., Yeungnam (2002), pp. 9–16.
6. S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity*, McGraw-Hill, New York (1970).
7. G. L. Gorynin and Yu. V. Nemirovskii, "Boundary layer in a layered rod," *Nauch. Vestn. Novosib. Gos. Tekh. Univ.*, No. 1(16), 21–36 (2004).
8. R. M. Rappoport, "Calculation of beams composed of materials with different mechanical characteristics," *Tr. Leningrad. Politekh. Inst.*, No. 5, 52–74 (1948).